

On predictive distributions for exchangeable sequences related to exponentially tilted Poisson-Kingman models.¹

Sulle distribuzioni predittive per successioni scambiabili relative ai modelli Poisson-Kingman ampliati esponenzialmente.

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Riassunto: Mostriamo che la successioni scambiabili guidate da una misura di probabilità aleatoria quasi certamente discreta, con atomi aleatori ordinati in senso decrescente con distribuzione Poisson-Kingman derivata da un subordinatore α -stabile ampliato esponenzialmente, sono le uniche con successione delle predittive in forma Gibbs, nella classe delle successioni scambiabili guidate da misure aleatorie normalizzate.

Keywords: Exchangeable partitions, Exponential Tilting, Poisson-Kingman models.

It is well known that given a law Q on the space $\mathcal{P}_1^\downarrow$ of decreasing sequences of positive numbers with sum 1, and a law $H(\cdot)$ on a Polish space $(S, \mathcal{S})$, a *random discrete probability measure* (RDPM) P on $\mathcal{S}$ may be defined by $P(\cdot) = \sum_{i=1}^\infty P_i \delta_{X_i}(\cdot)$, for X_i iid $\sim H(\cdot)$ and $(P_i) \sim Q$. Pitman (2003) introduces a large class of RDPMs, termed Poisson-Kingman models, deriving the law Q by a random discrete distribution $(P_i) = (J_i/T)$, where $J_1 \geq J_2 \geq \dots \geq 0$ are the random lengths of the ranked points of a Poisson process with Lévy density ρ , and $T = \sum_i J_i$. The family of basic Poisson-Kingman models is then enlarged to *mixed* models $PK(\rho, \gamma) := \int_0^\infty PK(\rho|t)\gamma(dt)$, where $PK(\rho|t)$ is the regular conditional distribution of (P_i) given $(T = t)$ constructed above, and γ is a probability distribution on $(0, \infty)$. Pitman (see Section 4.2) also states that: a *basic* model equivalent to a *mixed* model exists *if and only if* the basic model is driven by the *exponentially tilted* version of the Lévy density of the mixed model, and the mixing density is the corresponding tilted density. More formally: $PK(\rho_2) = PK(\rho_1, \gamma)$ for some γ if and only if $\rho_2(t) = \rho_1(t) \exp\{-\lambda t\}$ and $\gamma(t) = f_T(t) \exp\{\psi(\lambda) - \lambda t\}$, for every t , for some $\lambda > 0$, where f_T is the probability density corresponding to ρ_1 and $\psi(\lambda)$ is the Laplace exponent of f_T . Since it is well-known that only RDPMs whose ranked atoms follow a *basic* PK model admit corresponding equivalent construction in terms of normalization of completely random measures, an easy consequence of Pitman's results is that: a) *in the class of mixed PK models, only those obtained by exponential tilting share this property.*

Now recall that sampling from a RDPM induces an exchangeable random partition Π of the positive integers N , i.e. for each restriction $\Pi_n = \{A_1, \dots, A_k\}$ of Π to $[n] = \{1, \dots, n\}$, and for each n , $Pr(\Pi_n = \{A_1, \dots, A_k\}) = p(n_1, \dots, n_k)$, where $n_j = |A_j| \geq 1$ and $\sum_{j=1}^k n_j = n$, for some non-negative symmetric function p of finite sequences of positive integers called the *exchangeable partition probability function* (EPPF) determined by Π . By construction (see Hansen and Pitman, 2000) both basic and

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mixed PK models all induce exchangeable sequences (X_n) characterized by prediction rules of the form

$$Pr(X_{n+1} \in \cdot | X_1, \dots, X_n) = \sum_{j=1}^{k_n} p_{j,n} \delta_{X_j^*}(\cdot) + q_n H(\cdot) \quad (1)$$

where X_j^* , for $1 \leq j \leq k_n$ are distinct values, in order of appearance, in $(X_1, \dots, X_n)$, $H(\cdot)$ is a non atomic probability measure, $p_{j,n} = p(\mathbf{n}^{j+})/p(\mathbf{n})$ and $q_n = p(\mathbf{n}^{k+1})/p(\mathbf{n})$, where $p(\mathbf{n}^{j+}) := p(n_1, \dots, n_j + 1, \dots, n_k)$ and $p(\mathbf{n}^{k+1}) := p(n_1, \dots, n_k, 1)$. This implies that exchangeable sequences driven by PK models are completely identified by corresponding EPPFs. A particularly tractable case of (1) arises when the EPPF is of *Gibbs form* i.e.

$$p(n_1, \dots, n_k) = V_{n,k} \prod_{j=1}^k W_j$$

for some nonnegative weights $W = (W_j)$ and $V = (V_{n,k})$, for all $1 \leq k \leq n$ and all compositions $(n_1, \dots, n_k)$ of n . In a recent paper Gneden and Pitman (2005) provide a detailed analysis of this case, and show that, to be (Π_n) consistent as n varies, the weights W must be of the form $W_{n_j} = (1 - \alpha)_{n_j-1}$, depending on a single parameter $\alpha \in (-\infty, 1)$, and the weights V must be the solution to the backward recursion $V_{n,k} = (n - \alpha k) V_{n+1,k} + V_{n+1,k+1}$ with $V_{1,1} = 1$. In particular, for $\alpha \in (0, 1)$, they identify the extreme points of the infinite dimensional simplex of the possible weights $V_{n,k}$, as the extreme partitions generated by $PK(\rho_\alpha|t)$ models, with $t \in [0, \infty)$, for ρ_α the Lévy density of the α -stable subordinator. In other words they establish that: b) for $\alpha \in (0, 1)$, *exchangeable partitions in Gibbs form characterize mixed Poisson-Kingman models derived from the α -stable subordinator*.

Combining statements a) and b) it follows that, in the family of RDPMs obtained by normalization of completely random measures, only those whose ranked atoms follow *exponentially tilted* PK distributions derived by the α -stable subordinator for $\alpha \in [0, 1)$, induce exchangeable sequences with predictive distributions in Gibbs form. The explicit form of the EPPF for this class has been derived in Cerquetti (2006) exploiting results in Pitman (2003). The predictive distributions turn out to be of the form (1) for

$$p_{j,n}(\mathbf{n}) = \frac{2}{n} \frac{\eta_{n+1,k}}{\eta_{n,k}} (n_j - \alpha) \quad \text{and} \quad q_n(\mathbf{n}) = \frac{2}{n} \frac{\eta_{n+1,k+1}}{\eta_{n,k}} \alpha \delta \quad (2)$$

where $\eta_{n,k} = \int_0^\infty \lambda^{n-1} \frac{e^{-\delta(\gamma \frac{1}{\alpha} + 2\lambda)^\alpha}}{(\gamma \frac{1}{\alpha} + 2\lambda)^{n-k\alpha}} d\lambda$, and δ is the scale parameter of the stable distribution. Notice that, for $\alpha = 0$, the well-known Blackwell-MacQueen scheme is recovered.

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